1 From Part 1: Markov chain $X_0, X_1, \dots$ on a finite state space $\mathcal{V}$ (of size N), transition probabilities $P(x,y)$. Assume irreducible, aperiodic and time reversible: $\pi(x)P(x,y) = \pi(y)P(y,x)$, with π the stationary distribution. Inner product $\langle f, g\rangle_{\boldsymbol{\pi}} = \sum_{x\in\mathcal{V}} f(x)\pi(x)g(x)$. Time reversible implies self-adjoint: $\langle f, Pg\rangle_{\boldsymbol{\pi}} = \langle Pf, g\rangle_{\boldsymbol{\pi}}$. Spectrum of P:

$$\lambda_1 = 1 > \lambda_2 = 1-\gamma \geq \dots \geq \lambda_N > -1.$$

2 Given $f : \mathcal{V} \to \mathbb{R}$ with $\max_{x\in\mathcal{V}} |f(x)| \leq 1$ and $\sum_{x\in\mathcal{V}} \pi(x)f(x) = 0$ and $\sum_{x\in\mathcal{V}} \pi(x)f(x)^2 \leq \sigma^2$. Seek concentration of $t_n(f) = f(X_1) + \dots + f(X_n)$ under $\mathbb{P}_\pi$. Claim:

$$\mathbb{P}_\pi e^{\theta t_n(f)} \leq \text{const} \times \exp\left(\frac{W\theta^2}{2(1-B\theta)}\right) \qquad \text{for } 0 \leq \theta < 1/B$$

where $W = Cn\sigma^2/\gamma$ and $B = 5/\gamma$, leading to tail bound

$$\mathbb{P}_\pi\{t_n(f) \geq r\} \leq \text{const} \times \exp\left(-\frac{\gamma r^2/2}{Cn\sigma^2 + 5r}\right) \qquad \text{for } r \geq 0.$$

3 Matrix notation.

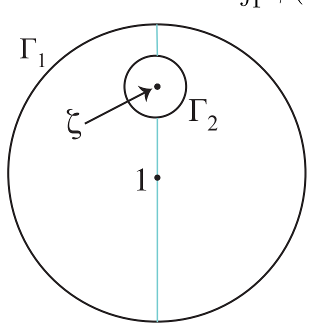
$$\begin{aligned}
&\pi \text{ an } N\times 1 \text{ column vector;} \qquad \Pi = \operatorname{diag}(\pi) \text{ an } N\times N \text{ matrix}\\
&P \text{ an } N\times N \text{ matrix with } \pi^\intercal P = \pi^\intercal \text{ and } P\mathbb{1} = \mathbb{1}\\
&f \text{ an } N\times 1 \text{ column vector;} \qquad F = \operatorname{diag}(f) \text{ an } N\times N \text{ matrix}\\
&P_\theta = Pe^{\theta F} \text{ and } T_\theta = e^{\theta F/2}Pe^{\theta F/2}
\end{aligned}$$

Note $\langle g, h\rangle_{\boldsymbol{\pi}} = g^\intercal \Pi h$ and $\pi^\intercal h = \mathbb{1}^\intercal \Pi h = \langle \mathbb{1}, h\rangle_{\boldsymbol{\pi}}$. Claim: $e^{-\theta F/2}$ and T_θ are self-adjoint and

$$\begin{aligned}
\mathbb{P}_\pi e^{\theta t_n(f)} &= \pi^\intercal P e^{\theta F} \dots P e^{\theta F} P e^{\theta f} = \pi^\intercal P_\theta^n \mathbb{1}\\
&= \pi^\intercal e^{-\theta F/2} T_\theta^n e^{\theta F/2}\mathbb{1}\\
&= \langle \mathbb{1}, e^{-\theta F/2} T_\theta^n e^{\theta f/2}\rangle_{\boldsymbol{\pi}} = \langle e^{-\theta f/2}, T_\theta^n e^{\theta f/2}\rangle_{\boldsymbol{\pi}}\\
&\leq \|e^{-\theta f/2}\|_{\boldsymbol{\pi}} \, |||T_\theta|||_\pi^n \, \|e^{\theta f/2}\|_{\boldsymbol{\pi}} \qquad \text{where } |||\cdot|||_\pi \text{ is operator norm}\\
&\leq e^\theta \lambda_1(\theta)^n \qquad \text{where } \lambda_1(\theta) \text{ is largest eigenvalue of } T_\theta.
\end{aligned}$$

The power could be reduced to $n-1$ by a slightly trickier argument.

4 Part 2: Complex analysis facts. If $\psi : G \to \mathbb{C}$ is holomorphic on a convex, open subset G of $\mathbb{C}$ and Γ is a closed path lying inside G then $\oint_\Gamma \psi(z)\,dz = 0$.



As a consequence: If $\Gamma_1(t) = 1 + \rho e^{it}$ and $\Gamma_2(t) = \zeta + \delta e^{it}$ for $0 \leq t \leq 2\pi$ with $|\zeta - 1| < \rho - \delta$ then

$$\oint_{\Gamma_1} \frac{1}{z-\zeta}dz = \oint_{\Gamma_2} \frac{1}{z-\zeta}dz = \int_0^{2\pi} \frac{i\delta e^{it}}{\delta e^{it}}\,dt = 2\pi i$$

but $\oint_{\Gamma_1}(z-\zeta)^{-k}dz = 0$ for $k \in \mathbb{Z}\backslash\{1\}$.

5 Let T be an $N \times N$ matrix. The resolvent $R_T(z) := (T - zI_N)^{-1}$ is well defined except at the eigenvalues of T. If T is self-adjoint (for $\langle \cdot, \cdot \rangle_{\boldsymbol{\pi}}$) there exists an orthonormal basis $w_1, \dots, w_N$ with $Tw_j = \mu_j w_j$. If $W = (w_1, \dots, w_n)$ then $W^\intercal \Pi W = I_N$ and $TW = W \text{diag}(\mu_1, \dots, \mu_N)$ and

Note $W^{-1} = W^\intercal \Pi$.

$$(\mu_1 - z) W^{-1} R_T(z) W = \text{diag}(1, 0, \dots, 0) + \text{diag}\left(0, \dots \frac{\mu_1 - z}{\mu_j - z}, \dots\right)$$

The jth diagonal element in the last matrix expands to

$$-\sum\nolimits_{k \geq 1} (\mu_1 - z)^k (\mu_1 - \mu_j)^{-k} \qquad \text{for } |\mu_1 - z| < \delta := \min_{j \geq 2} |\mu_1 - \mu_j|.$$

In matrix form,

$$(\mu_1 - z) R_T(z) = H_1 - \sum\nolimits_{k \geq 1} (\mu_1 - z)^k S^k$$

where $H_1 = w_1 w_1^\intercal \Pi$ is the matrix for orthogonal projection onto the subspace spanned by w_1 and

$$S := W \text{diag}\left(0, (\mu_1 - \mu_2)^{-1}, \dots, (\mu_1 - \mu_N)^{-1}\right) W^\intercal \Pi$$

is self-adjoint (for $\langle \cdot, \cdot \rangle_{\boldsymbol{\pi}}$) with $|||S|||_\pi \leq 1/\delta$. For the special case where T equals P we have $H_1 = \mathbb{1}\pi^\intercal$.
If $\Gamma(t) = 1 + \rho e^{it}$ for $0 \leq t \leq 2\pi$ and $|1 - \mu_1| < \rho < \min_{j \geq 2} |1 - \mu_j|$ then

$$\frac{1}{2\pi i} \oint_\Gamma R_T(z)\, dz = -H_1 \quad \text{AND} \quad \frac{1}{2\pi i} \oint_\Gamma (1 - z) R_T(z)\, dz = (\mu_1 - 1) H_1$$

Take the trace of the second equality to get a representation for $\mu_1 - 1$.

6 Think of the T_θ spectrum, $\lambda_1(\theta) \geq \lambda_2(\theta) \geq \dots \geq \lambda_N(\theta)$, as a perturbation of the spectrum of P. If $\Gamma(t) = 1 + \rho e^{it}$ for $0 \leq t \leq 2\pi$ and θ is small enough that

$$|1 - \lambda_1(\theta)| < \rho < \min_{j \geq 2} |1 - \lambda_j(\theta)|$$

then, by trace trickery,

$$\lambda_1(\theta) - 1 = \frac{1}{2\pi i} \oint_\Gamma (1 - z) \text{trace}(T_\theta - zI_N)^{-1} dz = \frac{1}{2\pi i} \oint_\Gamma (1 - z) \text{trace}(P_\theta - zI_N)^{-1} dz$$

7 Write $R_\theta(z)$ for $(P_\theta - zI_N)^{-1}$ and abbreviate $R_0(z)$ to $R(z)$. Write $P_\theta(z)$ as $P + A_\theta(z)$ where $A_\theta = \sum_{k \geq 1} \theta^k P F^k / k!$. If $|||A_\theta R(z)|||_\pi < 1$ then

$$R_\theta(z) = R(z) \left(I_N + A_\theta R(z)\right)^{-1} = R(z) + \sum\nolimits_{q \geq 1} (-1)^q R(z) \left(A_\theta R(z)\right)^q.$$